OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2604

Pure Mathematics 4

Wednesday **14 JANUARY 2004** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

1 A curve has equation $y = \dfrac{x-9}{(x+7)(x-5)}$.

(i) Write down the equations of the three asymptotes. [2]

(ii) Find $\dfrac{dy}{dx}$. Hence find the coordinates of the stationary points. [5]

(iii) Sketch the curve. [3]

(iv) Solve the inequality $\dfrac{x-9}{(x+7)(x-5)} > -\dfrac{1}{45}$. [5]

(v) On a separate diagram, sketch the curve with equation $y^2 = \dfrac{x-9}{(x+7)(x-5)}$.

Give the coordinates of the points on this curve where the tangent is parallel to the x-axis, and the point where the tangent is parallel to the y-axis. [5]

2 **(a)** Find $\sum_{r=1}^{n} r(3r-1)$, giving your answer in a fully factorised form. [5]

(b) **(i)** On an Argand diagram, mark the points A and B representing the complex numbers $0+9\mathrm{j}$ and $12+0\mathrm{j}$ respectively.

Describe in words the locus L_1 of points representing complex numbers w which satisfy $|w-9\mathrm{j}| = |w-12|$.

Draw L_1 on your diagram. [5]

The locus L_2 consists of the points representing complex numbers z for which $|z-9\mathrm{j}| = 2|z-12|$.

(ii) By writing $z = x + y\mathrm{j}$, where x and y are real, obtain an equation relating x and y, and hence show that L_2 is a circle. Give the centre and the radius of this circle. [8]

(iii) Hence write down an equation for L_2 in which z occurs only once. [2]

3 **(a)** Prove by induction that $\sum_{r=1}^{n} (3r-1)(r+1)4^r = n^2 4^{n+1}$. [7]

(b) Two straight lines L and M have equations

$$L: \frac{x-6}{5} = \frac{y+10}{-1} = \frac{z-15}{2} \quad \text{and} \quad M: \frac{x-7}{3} = \frac{y+5}{2} = \frac{z+8}{9}.$$

(i) Determine whether L and M intersect or not. [6]

(ii) Verify that the straight line N with equation $\frac{x-6}{1} = \frac{y+10}{3} = \frac{z-15}{-1}$ is perpendicular to both L and M. [2]

(iii) Find, in the form $ax + by + cz + d = 0$, the equation of the plane which contains L and N. [5]

4 Let $\mathbf{P} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} e & 0 & -b \\ 0 & f & 0 \\ -d & 0 & a \end{pmatrix}$.

(i) Find **PQ**. [4]

(ii) Given that **PQ** is a non-zero multiple of the identity matrix, express f in terms of a, b, c, d and e, and state any necessary conditions on a, b, c, d and e. [4]

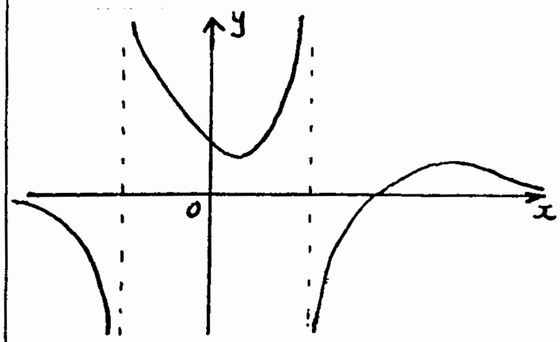
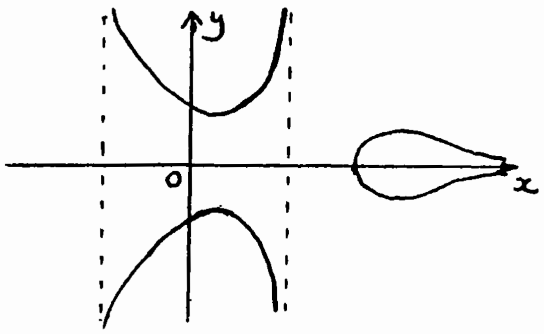
(iii) Find $\mathbf{P}^{-1}$, assuming that the conditions you stated in part **(ii)** are satisfied. [4]

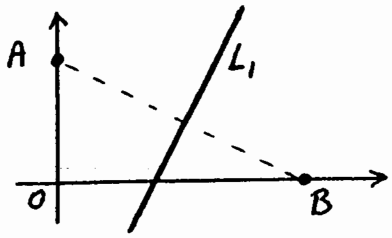
$\mathbf{M} = \begin{pmatrix} 3 & 0 & 8 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ and **N** is a 3×3 matrix with inverse $\mathbf{N}^{-1} = \begin{pmatrix} 1 & -2 & 3 \\ 3 & k & 0 \\ 2 & 4 & k \end{pmatrix}$.

(iv) Find $(\mathbf{MN})^{-1}$. [5]

(v) Given that $\mathbf{MN}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2k \\ 0 \end{pmatrix}$, express x, y and z in terms of k. [3]

Mark Scheme

1 (i)	$x=-7,\ x=5$ $y=0$	B1 B1 2	 NOT awarded for $y \to 0$
(ii)	$\frac{dy}{dx}=\frac{(x^2+2x-35)-(x-9)(2x+2)}{(x+7)^2(x-5)^2}$ $=\frac{-x^2+18x-17}{(x+7)^2(x-5)^2}$ For stationary points, $-x^2+18x-17=0$ $x=1,\ 17$ Stationary points are $(1,\ \frac{1}{4}),\ (17,\ \frac{1}{36})$	M1 A1 M1 A1A1 5	Use of quotient rule (or equivalent) Any correct form Give A1 for $x=1,\ 17$ *Accept* $(17,\ 0.028)$ *or better*
(iii)		B1 B1 B1 3	LH section: negative gradient, below x-axis Middle section: above x-axis, single minimum to right of y-axis *(Allow to left ft if stationary point wrong)* RH section: correct shape and asymptotic behaviour
(iv)	$\frac{x-9}{(x+7)(x-5)}=-\frac{1}{45}$ when $x^2+47x-440=0$ $x=-55,\ 8$ $\frac{x-9}{(x+7)(x-5)}>-\frac{1}{45}$ when $x<-55,\ \ -7<x<5,\ \ x>8$	M1 A1 M1 A2 5	Or first step in solving inequality Or factors $(x+55)(x-8)$ If M0, B1B1 for $-55,\ 8$ (or factors) Considering intervals defined by four critical values $-55,\ -7,\ 5,\ 8$ (ft) Give A1 if one minor error (e.g. $\le$) or give A1 ft if $-55,\ 8$ wrong (provided one is <-7, other is >5)
(v)	 Parallel to x-axis at $(1,\ \frac{1}{2}),\ (1,\ -\frac{1}{2})$ $(17,\ \frac{1}{6}),\ (17,\ -\frac{1}{6})$ Parallel to y-axis at $(9,\ 0)$	B2 B2 ft B1 5	Give B1 for 4 'sections' correct Give B1 ft if original graph wrong Give B1 for two correct

2 (a)	$\sum_{r=1}^{n}(3r^2 - r)$ $= \frac{1}{2}n(n+1)(2n+1) - \frac{1}{2}n(n+1)$ $= \frac{1}{2}n(n+1)(2n+1-1)$ $= n^2(n+1)$	M1 A1A1 M1 A1 **5**	Multiplying out and using formula for $\sum r^2$ or $\sum r$ Common factor $n(n+1)$, or simplified cubic
(b)(i)	 L_1 is the perpendicular bisector of AB	B1 B1 B1 B2 **5**	For A For B For L_1 drawn Or equivalent Give B1 for 'set of points which are equidistant from A and B' etc Give B1 for 'line bisecting A and B' Give B1 for 'a perpendicular bisector'
(ii)	$\lvert x + (y-9)\mathrm{j} \rvert = 2\lvert (x-12) + y\mathrm{j} \rvert$ $\sqrt{x^2 + (y-9)^2} = 2\sqrt{(x-12)^2 + y^2}$ $x^2 + (y-9)^2 = 4\{(x-12)^2 + y^2\}$ $3x^2 + 3y^2 - 96x + 18y + 495 = 0$ $x^2 + y^2 - 32x + 6y + 165 = 0$ $(x-16)^2 + (y+3)^2 = 100$ Centre $(16, -3)$ Radius 10	M1 A1 M1 M1 A1 M1 A1 A1 **8**	Finding one modulus Squaring *Dependent on first M1* Obtaining standard circle equation *Dependent on first M1* or other method for centre or radius Or $16 - 3\mathrm{j}$
(iii)	$\lvert z - (16 - 3\mathrm{j}) \rvert = 10$	B2 ft **2**	Give B1 ft for $\lvert z \pm (16-3\mathrm{j}) \rvert = \ldots$ or $\lvert z \pm \ldots \rvert = 10$ Max B1 ft if their centre is O

3 (a)	When $n=1$, LHS $=2\times2\times4=16$		
	RHS $=1\times4^2=16$	B1	
	Assuming true for $n=k$		
	$\sum_1^{k+1} = k^2 4^{k+1} + (3k+2)(k+2)4^{k+1}$	M1A2	
	$= 4^{k+1}\left\{k^2+3k^2+8k+4\right\}$		
	$= 4^{k+1}4(k^2+2k+1) = 4^{k+2}(k^2+2k+1)$	M1	Obtaining $4^{k+2}\times$ a quadratic *Dependent on previous M1*
	$=(k+1)^2 4^{k+2}$	A1	Correctly obtained
	True for $n=k \Rightarrow$ True for $n=k+1$	A1	*Dependent on previous M1A2M1A1*
	Hence true for all positive integers n	7	
(b)(i)	If they intersect, $6+5\lambda=7+3\mu$ (1)	M1	Equating two components, *using different parameters*
	$-10-\lambda=-5+2\mu$ (2)	A1	Two equations correct
	$15+2\lambda=-8+9\mu$ (3)	M1	Solving to obtain one of λ, μ
	Solving (1) and (2), $\lambda=-1$, $\mu=-2$	A1	[(2) and (3) $\Rightarrow \lambda=-7$, $\mu=1$ (1) and (3) $\Rightarrow \lambda=2$, $\mu=3$]
	Checking in (3), LHS $=13$, RHS $=-26$	M1	
	So L, M do NOT intersect	A1 6	Correct working only
	OR Solving $\frac{x-6}{5}=\frac{y+10}{-1}$ and $\frac{x-7}{3}=\frac{y+5}{2}$		
	gives $x=1$ ($y=-9$) M1A1		
	In L, $x=1 \Rightarrow z=13$ M1		
	In M, $x=1 \Rightarrow z=-26$ M1A1		
	Hence lines do NOT intersect A1		Correct working only
	OR $\left[\begin{pmatrix}5\\-1\\2\end{pmatrix}\times\begin{pmatrix}3\\2\\9\end{pmatrix}\right]\cdot\left[\begin{pmatrix}6\\-10\\15\end{pmatrix}-\begin{pmatrix}7\\-5\\-8\end{pmatrix}\right]$ M3		
	$=\begin{pmatrix}-13\\-39\\13\end{pmatrix}\cdot\begin{pmatrix}-1\\-5\\23\end{pmatrix}$ $(=507)\neq0$ A1 A1		For vector product For result is non-zero
	so lines do NOT intersect A1		Correct working only
(ii)	$\begin{pmatrix}1\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}5\\-1\\2\end{pmatrix}=5-3-2=0$ and $\begin{pmatrix}1\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\2\\9\end{pmatrix}=3+6-9=0$	B1B1 2	For other methods (e.g. finding vector product), give M1 for correct method A1 for completion
(iii)	$\begin{pmatrix}5\\-1\\2\end{pmatrix}\times\begin{pmatrix}1\\3\\-1\end{pmatrix}=\begin{pmatrix}-5\\7\\16\end{pmatrix}$	M1 A1	Vector product, or other method for finding normal vector
	Equation is $-5x+7y+16z=-30-70+240$	M1 M1	For $-5x+7y+16z$ *Dep on first M1* Using a point to find the constant *Indep*
	$5x-7y-16z+140=0$	A1 5	
	OR $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\-10\\15\end{pmatrix}+\lambda\begin{pmatrix}5\\-1\\2\end{pmatrix}+\mu\begin{pmatrix}1\\3\\-1\end{pmatrix}$		
	M3 $5x-7y-16z+140=0$ A2		Obtaining equation in x, y, z Give A1 for $5x-7y-16z$

4(i)	$\mathbf{PQ} = \begin{pmatrix} ae-bd & 0 & 0 \\ 0 & cf & 0 \\ 0 & 0 & ae-bd \end{pmatrix}$	B1B1B1 B1 4	Diagonal elements Six zeros
(ii)	$cf = ae - bd$ $f = \dfrac{ae-bd}{c}$ Conditions $c \neq 0$ $ae - bd \neq 0$	M1 A1 B1 B1 4	
(iii)	When $f = \dfrac{ae-bd}{c}$, $\mathbf{PQ} = (ae-bd)\mathbf{I}$ $\mathbf{P}^{-1} = \dfrac{1}{ae-bd}\mathbf{Q}$ $= \begin{pmatrix} \frac{e}{ae-bd} & 0 & \frac{-b}{ae-bd} \\ 0 & \frac{1}{c} & 0 \\ \frac{-d}{ae-bd} & 0 & \frac{a}{ae-bd} \end{pmatrix}$	M1 M1 A2 4	*Independent of first M1* Give A1 for 3 non-zero elements correct *Dependent on at least M1*
(iv)	$\mathbf{M}^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}$ $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$ $= \frac{1}{4}\begin{pmatrix} 1 & -4 & 1 \\ 12 & 2k & -24 \\ 8-k & 8 & 3k-16 \end{pmatrix}$	M1 A1 M1 M1 A1 5	Applying result in (iii) to **M** (or finding inverse otherwise) Evaluating product of $\mathbf{N}^{-1}$ and $\mathbf{M}^{-1}$ (in either order)
(v)	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & -4 & 1 \\ 12 & 2k & -24 \\ 8-k & 8 & 3k-16 \end{pmatrix}\begin{pmatrix} 0 \\ 2k \\ 0 \end{pmatrix}$ $x = -2k,\ y = k^2,\ z = 4k$	M1 A2 ft 3	Give A1 ft for one correct

Examiner’s Report

2604 Pure Mathematics 4

General Comments

The marks on this paper were a bit lower than usual, but there were still many excellent scripts, with about 20% of candidates scoring 50 marks or more (out of 60). Questions 1 and 3 were attempted by almost every candidate, question 2 was slightly less popular, and question 4 was by far the least popular. The number of candidates answering all four questions was rather higher than usual.

Comments on Individual Questions

Question 1 (Curve sketching and Inequalities)

This question was well answered, with half the attempts scoring 15 marks or more (out of 20). In part (i), the vertical asymptotes were almost always given correctly, but the horizontal asymptote was frequently missing or incorrect, or sometimes given in the wrong form (for example, 'the x-axis', $y \to 0$ or $y = \frac{1}{x}$). Several candidates divided the denominator by the numerator and obtained an oblique asymptote $y = x + 11$. In part (ii) the differentiation was usually done correctly, but then sign errors when simplifying the numerator often led to incorrect stationary points. The curve sketching in part (iii) was generally good, with the right-hand branch causing the most difficulty. In part (iv) a variety of correct methods were used to solve the inequality; the most efficient seemed to be solving it first as an equation, then using the graph to write down the solution. Those who multiplied both sides by $(x + 7)^2(x - 5)^2$ often multiplied out all the brackets and were then unable to re-factorise the resulting quartic. The critical values –7 and 5 often did not appear in the solution. The square root graph in part (v) was well understood, although the coordinates of the points where the tangent is parallel to an axis were often omitted or wrong (usually forgetting to square root the y-values, and sometimes squaring them instead).

(i) $x = -7, x = 5, y = 0$; (ii) $\frac{dy}{dx} = \frac{-x^2+18x-17}{(x+7)^2(x-5)^2}$, $\left(1, \frac{1}{4}\right), \left(17, \frac{1}{36}\right)$;

(iv) $x < -55, -7 < x < 5, x > 8$; (v) $\left(1, \pm\frac{1}{2}\right), \left(17, \pm\frac{1}{6}\right), (9, 0)$.

Question 2 (Series and Complex Numbers)

This was the worst answered question. Although several candidates did answer it efficiently and confidently, half the attempts scored 10 marks or less. This unexpected difficulty has caused the lower marks on this paper, and it is probably the reason why so many candidates went on to answer all four questions.

The series in part (a) was usually summed correctly, and part (b)(i) was very often answered correctly. Candidates were expected to describe the locus as 'the perpendicular bisector of AB' (or equivalent) and not just as 'the set of points equidistant from A and B'. Very many candidates failed to score any marks in part (ii); this idea was clearly unfamiliar. A very common error was to apply the result $|z_1| = |z_2| \Leftrightarrow z_1^2 = z_2^2$, which is true for real numbers but not for complex numbers; unfortunately this does not lead to the equation of a circle. Some candidates did not understand what was required in part (iii). It was possible to earn both marks here provided that a centre and radius had been stated in part (ii), even if these were wrong.

(a) $n^2(n + 1)$; (b)(i) Perpendicular bisector of AB; (ii) Centre (16, –3), radius 10;
(iii) $|z - (16 - 3j)| = 10$.

Question 3 (Induction and Vectors)

This was the best answered question, with half the attempts scoring 16 marks or more, and about 20% of candidates scoring full marks. In part (a) the proof by induction was very often handled confidently. The most common error was to omit the sum of k terms from the inductive step, trying to show that the $(k + 1)^{st}$ term is equal to the sum of $(k + 1)$ terms. In part (b)(i), most candidates understood how to determine that the lines did not intersect, although many made arithmetic slips in their working. Part (b)(ii) was also answered well, either by showing that two scalar products are zero, or by finding the vector product of the directions of L and M and showing this to be parallel to N; however, some did not show enough detail in their working, and lost a mark. In part (b)(iii) a fairly common error was to assume that the normal vector was given by the direction of M.

(b)(i) Lines do not intersect; (iii) $5x - 7y - 16z + 140 = 0$.

Question 4 (Matrices)

This question was attempted by about 35% of the candidates, and the mean mark was about 11. Parts (i) and (ii) were often answered correctly. Although the method for finding the inverse matrix in part (iii) was generally understood, the answer given very often contained f. The inverse of **P** was sometimes used, as intended, to find the inverse of **M** in part (iv), but many were able to write down $\mathbf{M}^{-1}$ independently, presumably from a calculator. The inverse matrices were frequently multiplied in the wrong order. Part (v) was reasonably well understood, and full credit was given for correct working from any matrix stated as the answer to part (iv).

(i) $\begin{pmatrix} ae-bd & 0 & 0 \\ 0 & cf & 0 \\ 0 & 0 & ae-bd \end{pmatrix}$; (ii) $f = \frac{ae-bd}{c}$, $c \neq 0$, $ae-bd \neq 0$;

(iii) $\begin{pmatrix} \frac{e}{ae-bd} & 0 & \frac{-b}{ae-bd} \\ 0 & \frac{1}{c} & 0 \\ \frac{-d}{ae-bd} & 0 & \frac{a}{ae-bd} \end{pmatrix}$; (iv) $\frac{1}{4}\begin{pmatrix} 1 & -4 & 1 \\ 12 & 2k & -24 \\ 8-k & 8 & 3k-16 \end{pmatrix}$; (v) $x = -2k$, $y = k^2$, $z = 4k$.